An exposition of the

Theorem of the cube.

(Exposition inspired by A. Mihatsch)

Warning: Every scheme appearing in these notes is locally Noetherian. More general results are available by Noetherian approximation.

0) A straigthnening of Seesaw

We begin by a little precision on,

Theorem (Seesaw). Let $X \xrightarrow{p} S$ be a proper flat morphism with geometrically integral fibers. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then there is a closed subscheme $Z \hookrightarrow S$ such that any $T \to S$ factors through Z if and only if $\mathcal{L}_{|X_T} \cong p_T^* \mathcal{M}$ for an $\mathcal{M} \in \mathrm{Pic}(T)$.

We say that $\mathcal{L}_{|X_T}$ is <u>T-trivial</u>.

<u>Proof</u> Note that we are defing Z by an universal property. Then, wlg. $S = \operatorname{spec} R$ and we are free to perform further open localisations.

- We already know that:

$$Z^{top} = \{ s \in S \mid \mathcal{L}_{|X_s} \text{ is trivial} \}$$

is a closed subset of S.

We denoted it by Z^{top} because it necessarly is the underlying top. space of the scheme $Z \hookrightarrow S$ that we are searching for.

As on $S \setminus Z^{top}$, ϕ works, we are furtermore free to perform further open localisatiors around points $z \in Z^{top}$.

Some remarks.

i) By coh-b.-c. and $H^0(X(s), \mathcal{O}_{X(s)}) = k(s)$ we have $p_* \mathcal{O}_X = \mathcal{O}_S$. This is stable by base change $T \to S$.

ii) $\mathcal{L}_{X_T}$ is the pullback of a line bundle $\mathcal{M}$ on T if and only if $p_T^* p_{T*} \mathcal{L} \xrightarrow{\sim} \mathcal{L}$ and $p_* \mathcal{L}$ a l.b.

In this case $\mathcal{M} \cong p_{T*} \mathcal{L}$.

<u>Proof</u> $X_T \xrightarrow{p_T} T$ satisfies $p_{T*} \mathcal{O}_{X_T} = \mathcal{O}_T$ by above remark. Then locally on T $\mathcal{M} \cong \mathcal{O}_T$ and $p_* p^* \mathcal{O}_T = p_* \mathcal{O}_{X_T} = \mathcal{O}_T$ □

On the desired Z $p_* \mathcal{L}|_{X_Z}$ should be a line bundle and

$$p^* p_* \mathcal{L}|_{X_Z} \xrightarrow{\sim} \mathcal{L}|_{X_Z}.$$

If we can find a Z such that:
"if $\mathcal{L}|_{X_T}$ is T-trivial, then $T \longrightarrow Z$

and $\mathcal{L}|_{X_Z}$ is Z-trivial." Z would work.

By the "representing theorem" there is $K_0 \longrightarrow K_1$ proj R-modules s.t. $\forall\, R \to A$

$$p_* \mathcal{L}_{X|_A} = \ker (K_0|_A \to K_1|_A)$$

By setting $C = \operatorname{coker} (K_1^\vee \to K_0^\vee)$, we get:

$$p_* \mathcal{L}|_{X_A} = \operatorname{Hom}_R (C, A).$$

Note that for $z \in Z^{top}$:

$$C(z) \cong k(z)$$

Then locally around z

Key for scheme structure $\left[\; C \cong R/I \;\right.$ (Because generated by one element by Nak.)

Note that then $\forall R \to A$

$$\left[p_* \mathcal{L}|_{X_A} \text{ vanishes on } I \cdot A. \right]$$

In order for $p_* \mathcal{L}|_{X_A}$ to be a

e.b. $\Rightarrow$ $R/I \hookrightarrow A$ $\Rightarrow$ $\operatorname{Spec} A \to V(I)$.
$I \cdot A = 0$

Now $p^* p_* \mathcal{L}|_{X_{V(I)}} \to \mathcal{L}|_{X_{V(I)}}$

is locally around $x(z)$ a surjection of line bundles say for $x(z) \subseteq U$.

Setting $Z = V(I)$ locally on $S \setminus p(X - U)$ we get the claim. $\square$

(closed by p proper)

1) Picard functor.

The notion of "base triviality" is well understood in the following context.

Def. Let $X \to S$ proper flat + geom int fibers.

$$\mathrm{Pic}_{X/S} : \mathrm{Sch}_{/S}^{op} \longrightarrow \mathrm{Set}$$
$$T \longmapsto \mathrm{Pic}(X_T) / p_T^* \mathrm{Pic}(T)$$

is the Picard functor. ○

Theorem. $\mathrm{Pic}_{X/S}$ is representable.

Remark. Seesaw theorem exaclty says that $\mathrm{Pic}_{X/S}$ is separated.

For $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}_{X/S}(S)$ Seesaw says that the locus $E_{\mathcal{L}_1 = \mathcal{L}_2}$ defined on points

$$E_{\mathcal{L}_1=\mathcal{L}_2}(T) = \{ T \to S \mid \mathcal{L}_{1|X_T} \cong \mathcal{L}_{2|X_T} \text{ mod } \mathrm{Pic}(T) \}$$

is a closed-subscheme.

Moreover, for $\mathcal{L} \in \mathrm{Pic}_{X|S}$ the universal

$Z \hookrightarrow S$ in the straightening of

Seesaw is:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "\mathcal{L}"] \\
S \arrow[r, "\mathcal{O}_X"'] & \mathrm{Pic}_{X|S}
\end{tikzcd}$$

2) Theorem of the Cube

Theorem. Let $X, Y \to S$ proper flat geom int. fibers $x \in X(S)$, $y \in Y(S)$, S connected.

Let $\mathcal{L}$ be a l.b. st. $\exists s \in S$ with $\mathcal{L}|X_s \times_{k(s)} Y_s$ trivial. (1)

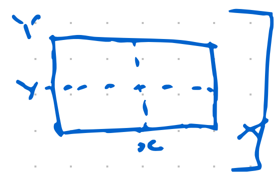
Then:

$$\mathcal{L} \cong p_X^* \mathcal{L}_{X \times \{y\}} \otimes p_Y^* \mathcal{L}_{\{x\} \times Y}$$

in $\mathrm{Pic}_{X \times_S Y/S}$ (i.e. mod $\mathrm{Pic}(S)$)

Precision: $p_X = X \times_S Y \to X \xrightarrow{\sim} X \times_S \{y\}$

$p_Y = X \times_S Y \to Y \xrightarrow{\sim} \{x\} \times_S Y$



Proof. By Seesaw, let $Z \hookrightarrow S$ be the closed subscheme where

$$\mathcal{M} = \mathcal{L}^{-1} \otimes p_X^* \mathcal{L}_{X \times \{y\}} \otimes p_Y^* \mathcal{L}_{\{x\} \times Y}$$

is Z-trivial.

By (1), $Z \neq \emptyset$. As S is connected it suffices to show: Z open.

Let $Z = V(I)$. Let $z \in Z$.

We want to show

$$\mathcal{O}_{S,z} \cdot I = 0$$

(Would show $\operatorname{spec}(\mathcal{O}_{S,z}) \to Z$, so that Z is open)

(Locally $S \cong \operatorname{spec} A$, $Z \cong \operatorname{spec}(A/I)$

$A \to B$, $A \to A/I$, $A/I \dashrightarrow B$ if $I \cdot B = 0$)

Enough to show:

$$\mathcal{O}_{S,z}/\mathfrak{m}_z^n \cdot I = 0 \qquad \forall n \geq 0.$$

$$\left(\Rightarrow I \subseteq \underbrace{\bigcap_n \mathfrak{m}_z^n}_{=0} \right)$$

To simplify the argument that comes we can replace $(\mathfrak{m}^n_z)$ by a sequence of ideals $(\mathfrak{a}_n)$ s.t

$$\mathfrak{a}_n/\mathfrak{a}_{n+1} \cong k(z)$$

and $\bigcap \mathfrak{a}_n = 0$.

Let $S_n = \operatorname{spec}\left(\mathcal{O}_{S,z}/\mathfrak{a}_n\right)$

$$V_n = S_n \times_S (X \times_S Y).$$

[Here we use the universal property of Z:

to show that

$$\mathcal{O}_{S,z}/\mathfrak{a}_n \cdot \tilde{I} = 0$$

we show that $\mathcal{M}|V_n$ is trivial.]

We have

$$\mathcal{O}_{V_1} \xrightarrow[\sim]{\theta_1} \mathcal{M}_1 \in H^0(V_1, \mathcal{M}_1) \left(\begin{array}{c} \text{As} \\ z \in Z \\ \text{by univ} \\ \text{prop of } Z \end{array} \right)$$

If we can lift θ_1 to

$$\mathcal{O}_{V_n} \xrightarrow{\theta_n} \mathcal{M}_n$$

θ_n would be an iso by NAK.

We have an exact sequence

flatness of $\mathcal{M}_n$

$$\begin{array}{c} 0 \to \mathfrak{a}_n \mathcal{M}_{n+1} \to \mathcal{M}_{n+1} \to \mathcal{M}_n \to 0 \\ \text{\rotatebox{90}{$\cong$}} \\ \mathcal{O}_{V_1} \end{array}$$

So:

$$H^0(V_{n+1}, \mathcal{U}_{n+1}) \longrightarrow H^0(V_n, \mathcal{U}_n) \xrightarrow{\delta} H^1(V_1, \mathcal{O}_{V_1})$$

And Θ_n lifts $\Leftrightarrow$ $\delta(\Theta_n) = 0$ by exactness of the sequence.

But by Künneth:

$$\begin{tikzcd}
H^0(V_n, \mathcal{U}_n) \arrow[r, "\delta"] \arrow[d] & H^1(V_1, G) \arrow[d, "\wr", "\text{Künneth iso}"'] \\
\begin{matrix} H^0((X \times \{y\})_n, \mathcal{U}_{n}|_{\cdots}) \\ \oplus \\ H^0((\{x\} \times Y)_n \times \mathcal{U}_{n}|_{\cdots}) \end{matrix} \arrow[r] & \begin{matrix} H^1((X \times \{y\})_1, G) \\ \oplus \\ H^1((\{x\} \times Y)_1, G) \end{matrix}
\end{tikzcd}$$

But $\delta(\Theta_{n}|_{(X \times \{y\})_n})$ and

$\delta(\Theta_{n}|_{(\{y\} \times X)_n})$ vanish, because $\Theta_n|(X \times \langle Y \rangle)_n$ and

$\Theta_n|(\langle Y \rangle \times X)_n$ lifts since

$\mathcal{M}|X \times \langle Y \rangle$ $\mathcal{M}_{\langle Y \rangle \times X}$ trivial mod $\mathrm{Pic}(S)$
by construction.

$\square$

Corollary. Let $X, Y_{/k=\bar{k}}$ proper and $Z_{/k=\bar{k}}$ connected. Then any l.b. $\mathcal{L} \in \mathrm{Pic}(X \times Y \times Z)$ is of form

$$p_{X\times Y}^{*} \mathcal{L}_1 \otimes p_{X\times Z}^{*} \mathcal{L}_2 \otimes p_{Y\times Z}^{*} \mathcal{L}_3$$

<u>Proof</u> Let $\mathcal{M} := \mathcal{L} \otimes p_{X\times Y}^{*} \mathcal{L}^{-1}_{|X\times Y\times\{z\}}$ for any $z \in Z(k)$. Then $\mathcal{M}$ satisfies the assertion of cube for

$$X \underset{Z}{\times} Y = X \times Y \times Z \to Z \qquad \square$$

Remark. Theorem of the cube

$$\Rightarrow \qquad \mathrm{Pic}^{\circ}_{X \underset{k}{\times} Y|k} \cong \mathrm{Pic}^{\circ}_{X|k} \times \mathrm{Pic}^{\circ}_{Y|k}$$

$\not\!Z$ False for Pic. (Already fails on product of ell. curves!)

• We prove the last remark, i.e. If $\exists x \in X(k), y \in Y(k)$

(Automatic if $k = \bar{k}$)

Theorem of the cube $\Rightarrow$ $\mathrm{Pic}^0_{X \times_k Y/k}$

$$\cong \mathrm{Pic}^0_{X/k} \times_k \mathrm{Pic}^0_{Y/k}$$

assuming $\mathrm{Pic}^0_{X/k}$ is representable.

$\mathrm{Pic}^0_{X \times_S Y/k}$ is a connected scheme.] Key.

(May be defined as conn. comp. of the identity of $\mathrm{Pic}_{X \times_k Y/k}$)

Let $\mathcal{P} \in \mathrm{Pic}^0_{X \times_k Y/k}(\mathrm{Pic}^0_{X \times_k Y/k})$

be the universal element.

$$\mathcal{P} \in \mathrm{Pic}((X \times_k Y/ \times_k \mathrm{Pic}^0_{X \times_k Y}) \big/ \mathrm{Pic}(\mathrm{Pic}^0_{X \times_k Y})$$

Key. By cube applied to

$$X \times_k Y \times_k \mathrm{Pic}^0_{X \times_k Y} \to \mathrm{Pic}^0_{X \times_k Y/k}$$

and $\mathcal{P}$.

Note that for

$$s = \mathcal{O}_{X \times_k Y} \in \mathrm{Pic}^0_{X \times Y / k}(k)$$

$\mathcal{P}(s) \cong \mathcal{O}_{X \times_k Y}$ so we can indeed apply cube.

Key.

We know therefore that

$$\mathcal{P} \cong (p_X, \mathrm{id})^* \mathcal{P}|_{X \times \{y\} \times \mathrm{Pic}^0_{X \times_k Y/k}}$$

$$\otimes (p_Y, \mathrm{id})^* \mathcal{P}|_{\{x\} \times Y \times \mathrm{Pic}^0_{X \times_k Y/k}}$$

mod $\mathrm{Pic}^0_{X \times_k Y/k}$

Consider now:

Rest is immediate.

$$\mathrm{Pic}^0_{X \times_k Y/k} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Pic}^0_{X/k} \times_k \mathrm{Pic}^0_{Y/k}$$

$$\mathcal{L} \longmapsto (\mathcal{L}|_{X \times \{y\}}, \mathcal{L}|_{\{x\} \times Y})$$

$$p_X^* \mathcal{L}_1 \otimes p_X^* \mathcal{L}_2 \longleftarrow (\mathcal{L}_1, \mathcal{L}_2)$$

with $\varphi \circ \psi = \mathrm{id}$.

We show that $\psi \circ \varphi = id.$

Let $T \in Sch_{/k}$. Let

$$\mathcal{L} \in \mathrm{Pic}^0_{X \times_k Y | k}(T) = \frac{\mathrm{Pic}^0(X \times_k Y \times_k T)}{p^* \mathrm{Pic}\, T}$$

and $f_{\mathcal{L}} : T \longrightarrow \mathrm{Pic}^0_{X \times_k Y | k}$

corresponding, so that

$$\mathcal{L} \cong_T (id, id, f_{\mathcal{L}})^* \mathcal{P}$$

But using the input from cube:

$$\mathcal{P} \cong (p_X, id)^* \mathcal{P}|_{X \times \{y\} \times \mathrm{Pic}^0_{X \times_k Y | k}}$$

$$\otimes (p_Y, id)^* \mathcal{P}|_{\{x\} \times Y \times \mathrm{Pic}^0_{X \times_k Y | k}}$$

$$\mod \mathrm{Pic}^0_{X \times_k Y | k}$$

We get: (See next page for precision if wanted.)

$$\mathcal{L} \cong_T p_X^* \mathcal{L}|_{X \times \{y\}} \otimes p_Y^* \mathcal{L}|_{\{x\} \times Y}$$

$\square$

To be really precise, last iso. follows from:

$$\begin{tikzcd}
X \times_{\mathbb{R}} Y \times_{\mathbb{R}} T \arrow[d, "{(id, id, f_g)}"] \arrow[dddd, bend right=60, "{(id, y, f_g)}"'] \arrow[dddr, bend left=40, "{(id, y, id)}"] & \\
X \times_{\mathbb{R}} Y \times_{\mathbb{R}} \mathrm{Pic}^0_{X \times_{\mathbb{R}} Y | \mathbb{R}} \arrow[d] & \\
X \times_{\mathbb{R}} \mathrm{Pic}^0_{X \times_{\mathbb{R}} Y | \mathbb{R}} \arrow[d, "\wr"] & \\
X \times_{\mathbb{R}} \{y\} \times_{\mathbb{R}} \mathrm{Pic}^0_{X \times_{\mathbb{R}} Y | \mathbb{R}} \arrow[d, phantom, "\wr"] & X \times_{\mathbb{R}} Y \times_{\mathbb{R}} T \arrow[dl, "{(id, id, f_g)}"] \\
X \times_{\mathbb{R}} Y \times_{\mathbb{R}} \mathrm{Pic}^0_{X \times_{\mathbb{R}} Y | \mathbb{R}} &
\end{tikzcd}$$

And similarly for x.

(Where $y : Y \longrightarrow Y$ denotes the constant function with value y.)